Lecture 4: First Properties of Schemes

Note Title 6/21/2019

Definition: A scheme X is ~~connected~~ irreducible if X is ~~connected~~ irreducible.

$\mathrm{Spec}(\prod_i A_i) = \prod_i \mathrm{Spec}(A_i)$. An affine is connected iff not product.

$\mathfrak{p} \lhd \prod_i A_i$ prime $\Rightarrow$ $\mathfrak{p} = A_1 \times \cdots \times \mathfrak{p}_i \times \cdots \times A_n$, $\mathfrak{p}_i \lhd A_i$ prime.

$\mathrm{Spec}\,A$ is irreducible iff $\mathrm{nil}(A)$ is prime. ex. $k[x,y]/(xy)$: (0) NOT prime

- $f \notin \mathrm{Nil}(A) = \bigcap_{\mathfrak{p}:\text{prime}} \mathfrak{p} \iff D(f) \neq \phi$
- $f_1, f_2 \notin \mathrm{Nil}(A)$, $\underset{\neq\phi}{D(f_1)} \cap \underset{\neq\phi}{D(f_2)} = D(f_1 f_2) = \phi$ iff $f_1 f_2 \in \mathrm{Nil}(A)$

Definition: A scheme X is reduced/integral if $\mathcal{O}_X(U)$ is reduced/integral, for every $U \underset{\text{open}}{\subseteq} X$

(reduced: no nilpotent element)

- $\mathcal{O}_X(U)$ has no nilpotent element, $\forall U \iff \mathcal{O}_{X,\mathfrak{p}}$ no nilpotent element, $\forall \mathfrak{p}$
- $X = \mathrm{Spec}\,A$ affine scheme

  X reduced iff $\mathrm{nil}(A) = 0$

  integral iff A integral domain

Proposition 1: A scheme is integral iff reduced & irreducible.

pf: $(\Rightarrow)$ If $X = U_1 \amalg U_2$ (open; NOT irreducible), then $\mathcal{O}_X(U_1 \amalg U_2) \cong \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ not integral

If $a \in \mathcal{O}_X(U)$ ($U \subseteq X$) nilpotent, $a^n = 0$ for some $n \in \mathbb{N}$ $\Rightarrow$ NOT integral

($\Leftarrow$) Now X reduced & irreducible

$\cup|$ open

U

$f_1 f_2 = 0 \in \mathcal{O}_X(U) \rightsquigarrow U_i = \{ p \in U \mid f_i \in m_p \}$ closed subset of U

$U_1 \cup U_2 = U \Longrightarrow \underline{U_1 = U}$ or $U_2 = U$

U irreducible

open subset of irreducible space is again irreducible

$\Downarrow$ f nilpotent iff $D(f) = \phi$ restrict to affine charts

f_1 nilpotent $\Longrightarrow f_1 = 0$

X reduced

Definition: • A scheme X is <u>locally Notherian</u> if $X = \bigcup_i \operatorname{Spec} A_i$, A_i Notherian.

• <u>Notherian</u> $=$ locally Notherian $+$ <u>quasi-compact</u>

every open cover has a finite subcover

X Notherian $\Longrightarrow$ $\underline{X}$ Notherian

ex. Every affine scheme is quasi-compact

ex. $\operatorname{Spec} k[x_1, x_2, \cdots]$ is NOT locally Noetherian

$X = \coprod A_i^1 / \sim$, $\quad A_i^1 \hookleftarrow \mathbb{G}_m \hookrightarrow A_j^1$

is locally Noetherian but NOT Noetherian.

Proposition 2: A scheme X is Notherian iff every affine open subset $U \cong \operatorname{Spec} A$, A is a Notherian ring.

pf: ($\Leftarrow$) obvious

($\Rightarrow$) WLOG assume $X = \operatorname{Spec} A \supseteq U = \operatorname{Spec} B$ (open)

$$\begin{matrix} A & \longrightarrow & B \\ f & & \bar{f} \end{matrix} \Rightarrow B_{\bar{f}} = A_f$$

localization of a Notherian ring is Notherian.

It suffices to prove that

If $f_i \in A$ (i finite) generate $1 \in A$, then A_{f_i} Notherian $\Rightarrow$ A Notherian

$\{\operatorname{Spec} A_{f_i}\}$ covers A i.e. $\bigcup_i D(f) = \operatorname{Spec} A \Leftrightarrow \bigcap V(f) = \phi$, $\bigcap V(f) = V(\Sigma(f_i))$

$\varphi_i : A \longrightarrow A_{f_i}$

Claim: $\mathfrak{a} \triangleleft A$, $\mathfrak{a} = \bigcap \varphi_i^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$

Now $\mathfrak{a}_i \triangleleft A$ ascending chain

$\varphi_i(\mathfrak{a}_j) \cdot A_{f_i}$ stabilizes since A_{f_i} is Notherian

Claim $\Longrightarrow$ $\mathfrak{a}_j$ stabilizes and A is Notherian.

proof of the claim:

- $\mathfrak{a} \subseteq \bigcap \varphi_i^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$
- $b \in \bigcap \varphi_i^{-1}(\varphi_i(\mathfrak{a}) \cdot A_{f_i})$

$$\varphi_i(b) = \frac{a_i}{f^{n_i}} \in A_{f_i}, \quad a_i \in \mathfrak{a}, \quad n_i \geq 0 \text{ (?)}, \quad \forall i$$

i finite, may assume $n_i = n$

$$\Longrightarrow \quad f_i^{m_i}(f^n b - a_i) = 0 \in A$$

again may assume $m_i = m_j$

$$\Longrightarrow \quad f_i^n b - a_i = 0 \text{ in } A \quad \text{or} \quad f_i^n b \in \mathfrak{a}$$

$$\left(1 = \sum_i c_i f_i\right)^{N \gg 0} \quad \therefore \quad b = \sum_i c_i f_i^n b \in \mathfrak{a}$$

$$\downarrow$$

$$1 = \sum_i c_i f_i^n$$

$f: X \to Y$ morphism of schemes

Definition. ① f <u>open immersion</u> if f induces an isomorphism of X w/ an open subscheme of Y

② f <u>closed immersion</u> if $X \underset{\text{top}}{\simeq} f(X)$, $\mathcal{O}_Y \twoheadrightarrow f_* \mathcal{O}_X$

ex. $\mathfrak{a} \triangleleft A$, then $\operatorname{Spec} A/\mathfrak{a} \hookrightarrow \operatorname{Spec} A$ closed immersion

this is the local model of closed immersion

In particular, $\mathfrak{a} = \operatorname{nil}(A)$, $\operatorname{Spec}(A/\mathfrak{a}) = (\operatorname{Spec} A)_{\text{red}}$

Definition:

- f is **of finite type** if $Y = \bigcup_i \operatorname{Spec} B_i$
  $f^{-1}(\operatorname{Spec} B_i) = \bigcup_{j \text{ finite}} \operatorname{Spec}(C_{ij})$, $C_{ij} = B_i$-algebra, finitely generated (quotient of polynomial ring)

  locally look like "varieties" & quasi-compact

- f is a **finite morphism** if $Y = \bigcup_i \operatorname{Spec} B_i$
  $f^{-1}(\operatorname{Spec} B_i) = \operatorname{Spec} A_i$, $A_i = B_i$-algebra & finite B_i-module

  $\Downarrow$

  f is closed & $f^{-1}(y)$ is finite

  Consequence of "going-up theorem"

  This reduces to the affine case $\operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B \ni \mathfrak{p}$

  $f^{-1}(\mathfrak{p}) = \operatorname{Spec} A \times_{\operatorname{Spec} B} \kappa(\mathfrak{p}) \cong \operatorname{Spec}\left(A \otimes_B \kappa(\mathfrak{p})\right)$, where $A \otimes_B \kappa(\mathfrak{p})$ is a finite dim'l $\kappa(\mathfrak{p})$-alg.

Fact: R = finite dim'l alg./$\mathbb{k}$, finitely generated

$\Longrightarrow$ R is Artinian

1. All prime ideals are maximal
2. only finitely many maximal ideal

ex. $\operatorname{Spec} \mathbb{R} \longrightarrow \operatorname{Spec} \mathbb{Q}$ is NOT of finite type

ex. $V(x - y^2) \longrightarrow \mathbb{A}^1_x$ is a finite morphism

$k[x,y]/(x-y^2) \longleftarrow k[x]$

$k[x,y]/(x-y^2) = k[x] + k[x]\cdot y$ as module

ex. Closed embedding are finite.